Fifth Annual Upper Peninsula
High School Math Challenge

Northern Michigan University (Marquette Co, MI)
Saturday 15 March 2014

TEAM: SOLUTION

SCHOOL: ______________________

TEAM PROBLEMS **TIME: 45 minutes**

1. 53,334
2. $a=3$ $b=\frac{2}{13}$
3. $\frac{\sqrt{21}}{4}$
4. $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ or $2+\sqrt{3}$
5. 6,643

Put no work on this side of the paper. Write the answers only in the above spaces. Put all work on the enclosed sheets of scrap paper, and hand in the scrap paper with your answer sheet.

1. Find the sum of the first 400 terms of the following series:

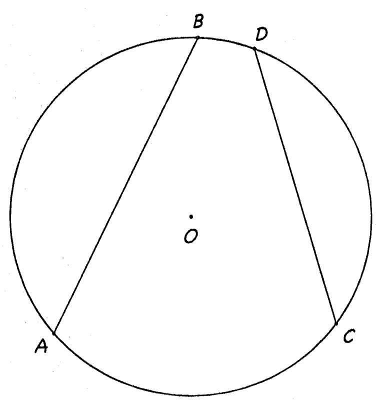
$1 + 3 - 5 + 7 + 9 - 11 + 13 + 15 - 17 + \ldots\ldots$

2. Let $f(x) = ax + b$. Find all real values of a and b such that

$$f(f(f(1))) = 29$$

and

$$f(f(f(0))) = 2.$$



3.

The diagram shows a circle, center O and radius 10.
$\overline{AB}$ and $\overline{CD}$ are chords of the circle.
The distance from O to $\overline{AB}$ is 4 units and the distance from O to $\overline{CD}$ is 6 units.
What is the ratio of the length of $\overline{AB}$ to the length of $\overline{CD}$?

4. A circle with radius 1 is tangent to both sides of a 60° angle. A second circle, larger than the first, is tangent to the first circle and to both sides of the angle. Find the radius r of the second circle.

5. If a, b, and c are integers such that $(a + b + c = 91)$ and $(a \bullet b \bullet c = 729)$, determine the value of $(a^2 + b^2 + c^2)$.

1. Break it up into 3 series:

$1 + 7 + 13 + 19 + \ldots$ (134 terms)

$3 + 9 + 15 + 21 + \ldots$ (133 terms)

$-(5 + 11 + 17 + 23 + \ldots)$ (133 terms)

The sum of an arithmetic progression is

$$S = \frac{n}{2}\left(2a + (n-1)d\right)$$

a is the first term
n is the number of terms
d is the difference between successive terms)

FIRST: $S = \frac{134}{2}\left(2(1) + 133 \cdot 6\right) = 67 \cdot 800 = 53{,}600$

SECOND: $S = \frac{133}{2}\left(2(3) + 132 \cdot 6\right) = 133 \cdot \frac{798}{2} = 133 \cdot 399 = 53{,}067$

THIRD: $S = \frac{133}{2}\left(2(5) + 132 \cdot 6\right) \quad 133 \cdot \frac{802}{2} = 133 \cdot 401 = 53{,}333$

$53{,}600 + 53{,}067 - 53{,}333 = 53{,}334$

2. $f(1) = a+b$

$f(f(1)) = a(a+b)+b = a^2+ab+b$

$f(f(f(1))) = a(a^2+ab+b)+b = a^3+a^2b+ab+b$

$f(0) = b$

$f(f(0)) = ab+b$

$f(f(f(0))) = a(ab+b)+b = a^2b+ab+b$

$f(f(f(1))) - f(f(f(0))) = a^3$

$29 - 2 = a^3$

$a^3 = 27$

$a = 3$

$f(f(f(0))) = 2$

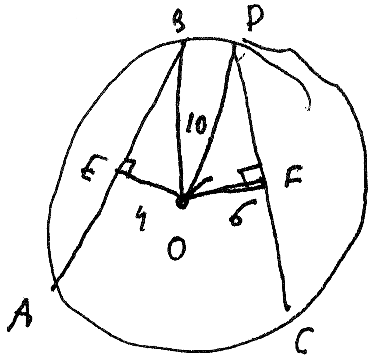
$3^2b+3b+b = 2$

$9b+3b+b = 2$

$13b = 2$

$b = 2/13$

3.



$OB = 10 \qquad OD = 10$

$OE = 4 \qquad OF = 6$

$EB^2 + OE^2 = OB^2$

$EB^2 + 16 = 100$

$EB^2 = 84$

~~$OB = 10$ $OD = 10$~~ $EB = \sqrt{84} = 2\sqrt{21}$

$OF^2 + DF^2 = OD^2$

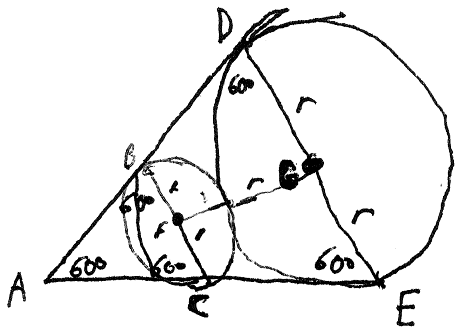
$6^2 + DF^2 = 100$

$DF^2 = 64$

$DF = 8$

$$\frac{AB}{CD} = \frac{EB}{DF} = \frac{2\sqrt{21}}{8} = \frac{\sqrt{21}}{4}$$

4.



By symmetry, $m\angle ABC = m\angle BCA = m\angle ADE = m\angle DEA = 60°$

$\triangle ABC$ and $\triangle ADE$ are equilateral.

The length of the altitude of an equilateral triangle of sides s is $\frac{s\sqrt{3}}{2}$

$\triangle ABC$ has $s=2$ so its altitude is $\frac{2\sqrt{3}}{2} = \sqrt{3}$

So, $\triangle ADE$ has $s = 2r$ and altitude of $\sqrt{3}+1+r$

$$\frac{2r\sqrt{3}}{2} = \sqrt{3}+1+r$$

$$r\sqrt{3} = \sqrt{3}+1+r$$

$$r(\sqrt{3}-1) = \sqrt{3}+1$$

$$r = \frac{\sqrt{3}+1}{\sqrt{3}-1} \cdot \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \frac{3+2\sqrt{3}+1}{3-1} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$$

5. $729 = 3^6$ so a, b, c are all powers of 3.

$a, b, c \not< 91$ so $a, b, c \leq 3^4 = 81$.

Trial and error gives $\{a, b, c\} = \{1, 9, 81\}$

$a^2 + b^2 + c^2 = 1 + 81 + 6561 = 6643$