# Fourth Annual Upper Peninsula
# High School Math Challenge

Northern Michigan University (Marquette Co, MI)

Saturday 23 March 2013

TEAM: SOLUTION

SCHOOL: ____________________

**TEAM PROBLEMS** **TIME: 45 minutes**

1. $(x-3)(x^2+3x+9)(x-2)$
2. $2/9$
3. $\frac{3+2\sqrt{3}}{3}$
4. $\frac{3\pi}{2}$
5. $\frac{1}{35}$

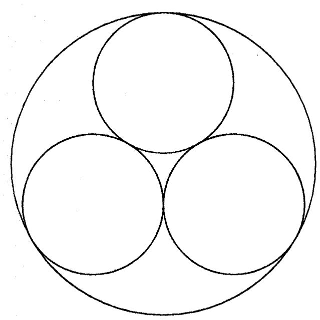
**Put no work on this side of the paper. Write the answers only in the above spaces. Put all work on the enclosed sheets of scrap paper, and hand in the scrap paper with your answer sheet.**

1. Completely factor: $x^4 - 2x^3 - 27x + 54$

2. Find all values of $x$ that satisfy the following equation:

$$\sqrt{2x+5} = 2\sqrt{2x} + 1$$

3. Three congruent circles are pairwise tangent to each other, and a larger circle is tangent to all three of the smaller circles. If the small circles have radius 1 inch, find, in inches, the radius of the large circle.



4. Find all solutions to the following equation in the interval $[0, 2\pi]$.

$$\cos^2 x + \sin x + 1 = 0$$

5. Four boys and three girls are seated, in a row, at random, to watch a play. What is the probability that boys and girls are seated alternately? Write your answer as a reduced fraction.

1. $x^4 - 2x^3 - 27x + 54$

$x^3(x-2) - 27(x-2)$

$(x^3-27)(x-2)$

$(x-3)(x^2+3x+9)(x-2)$

2. $\sqrt{2x+5} = 2\sqrt{2x} + 1$

$2x+5 = 4(2x) + 4\sqrt{2x} + 1$

$2x+5 = 8x + 4\sqrt{2x} + 1$

$-6x + 4 = 4\sqrt{2x}$

$36x^2 - 48x + 16 = 32x$

$36x^2 - 80x + 16 = 0$

$9x^2 - 20x + 4 = 0$

$9x^2 - 2x - 18x + 4 = 0$

$x(9x-2) - 2(9x-2) = 0$

$(x-2)(9x-2) = 0$

$x = 2 \quad x = 2/9$

$\sqrt{2(2)+5} = 2\sqrt{2(2)} + 1$

$\sqrt{9} = 2\sqrt{4} + 1$

$3 = 4 + 1$

$3 = 5$

DOES NOT CHECK

$\sqrt{2(\frac{2}{9})+5} = 2\sqrt{2(\frac{2}{9})} + 1$

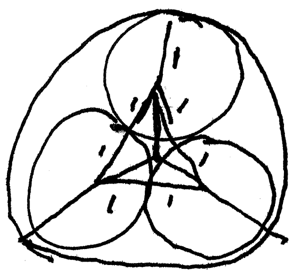
$\sqrt{\frac{4}{9}+\frac{45}{9}} = 2\sqrt{\frac{4}{9}} + 1$

$\sqrt{\frac{49}{9}} = 2(\frac{2}{3}) + 1$

$\frac{7}{3} = \frac{4}{3} + \frac{3}{3}$

$\frac{7}{3} = \frac{7}{3}$ checks

3,



The altitude of a triangle of sides s is given by $\frac{s\sqrt{3}}{2}$, so the altitude of the interior triangle is $\frac{2\sqrt{3}}{2} = \sqrt{3}$

The altitudes intersect $\frac{2}{3}$ of the way down so The distance between the center of the big circle and the center of each of the small circles is $\frac{2\sqrt{3}}{3}$

So the radius must be $1 + \frac{2\sqrt{3}}{3}$ or $\frac{3+2\sqrt{3}}{3}$

4,

$\cos^2 x + \sin x + 1 = 0$

$\cancel{(1-\cos^2 x}$

$(1-\sin^2 x) + \sin x + 1 = 0$

$-\sin^2 x + \sin x + 2 = 0$

Let $y = \sin x$

$-y^2 + y + 2 = 0$

$-y^2 + 2y - y + 2 = 0$

$-y(y-2) - 1(y-2) = 0$

$(y-2)(-y-1) = 0$

$y-2=0 \quad -y-1=0$

$y=2 \quad y=-1$

$\cancel{\sin x = 2}$

$\sin x = 2 \quad \sin x = -1$

sin x can't equal 2

$\sin x = -1$, $x = \frac{3\pi}{2}$

5. There are 4! ways to arrange the boys and 3! ways to arrange the girls. (And 7! ways to arrange everybody.)

$$\frac{4!\,3!}{7!} = \frac{4!\,3!}{7\cdot 6\cdot 5\cdot 4!} = \frac{3!}{7\cdot 6\cdot 5} = \frac{1}{35}$$