Third Annual Upper Peninsula High School Math Challenge

NORTHERN MICHIGAN UNIVERSITY (MARQUETTE, MI, USA)

Saturday 17 March 2012

NAME: SOLUTION

TEAM: _____

SCHOOL: _____

PROBLEM 1 **TIME: 4 minutes**

$x(2x-3)(3x+4)$

answer

Put no work on this side of the paper. Write the answer and only the answer in the space above. Put all work on the other side of the sheet.

Factor completely:

$6x^3 - x^2 - 12x$

$x(6x^2 - x - 12)$

$x(6x^2 + 8x - 9x - 12)$

$x(2x(3x+4) - 3(3x+4))$

$x(2x-3)(3x+4)$

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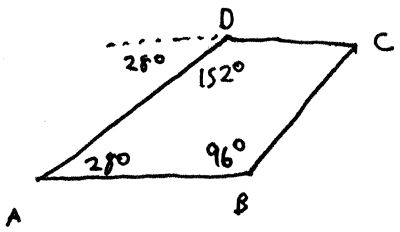
PROBLEM 2 **TIME: 3 minutes**

152°

answer

Put no work on this side of the paper. Write the answer and only the answer in the space above. Put all work on the other side of the sheet.

Trapezoid ABCD has $\overline{AB} \parallel \overline{DC}$. $m\angle A = 28°$. $m\angle B = 96°$. Find, in degrees, $m\angle D$.



Alternate interior angles are congruent and so

$m\angle D + 28° = 180°$

$m\angle D = 152°$

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NAME: SOLUTION

TEAM: _____

SCHOOL: _____

PROBLEM 3 **TIME: 3 minutes**

40

answer

Put no work on this side of the paper. Write the answer and only the answer in the space above. Put all work on the other side of the sheet.

My favorite restaurant offers an Early Bird Special.

I select one of the following entrees:
chicken fish steak portabello

I also select two of the following sides:
fries slaw applesauce baked beans mixed vegetables

How many possible Early Bird Specials are there?

There are 4 possible entrees, and I choose 1.

$$_4C_1 = \frac{4!}{1!(4-1)!} = \frac{4!}{3!} = 4$$

There are 5 possible sides, and I choose 2.

$$_5C_2 = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot 3!}{2!\,3!} = \frac{5 \cdot 4}{2} = 10$$

There are $4 \cdot 10 = 40$ possible Early Bird Specials.

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NAME: SOLUTION

TEAM: _____

SCHOOL: _____

PROBLEM 4 **TIME: 3 minutes**

75°

answer

Put no work on this side of the paper. Write the answer and only the answer in the space above. Put all work on the other side of the sheet.

When it is 3:30, what is the measure of the angle, in degrees, between the hour and minute hands of a standard clock?

There are 360° in a circle. The numbers divide it into 12 regions with $\frac{360^\circ}{12} = 30^\circ$ per region

At 3:30 the hour hand is halfway between 3 and 4 and the minute hand is on 6, so there are $2\frac{1}{2}$ regions between the hands,

$\frac{5}{2} \cdot 30 = 5 \cdot 15 = 75^\circ$

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Saturday 17 March 2012

NAME: SOLUTION

TEAM: _____

SCHOOL: _____

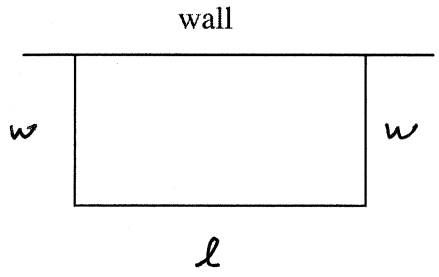
PROBLEM 5 **TIME: 4 minutes**

800 sq. ft.

answer

Put no work on this side of the paper. Write the answer and only the answer in the space above. Put all work on the other side of the sheet.

A rectangular pen is to be made using 80 feet of fencing. The fourth side will be a wall of the house. Find the maximum area, in square feet, that can be enclosed.



$l + 2w = 80$

$l = 80 - 2w$

$A = lw$

$A = (80 - 2w)w$

$A = 80w - 2w^2$

$A = -2w^2 + 80w$ is a parabola. The maximum point will be at the vertex: $w = -\frac{b}{2a} = -\frac{80}{2(-2)} = \frac{80}{4} = 20$

$l = 80 - 2 \cdot 20 = 80 - 40 = 40$

$A = lw = (40)(20) = 800$

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Saturday 17 March 2012

NAME: SOLUTION

TEAM: _____

SCHOOL: _____

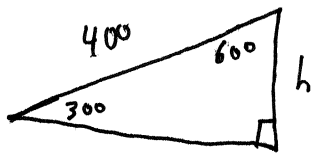
PROBLEM 6 **TIME: 4 minutes**

200 m

answer

Put no work on this side of the paper. Write the answer and only the answer in the space above. Put all work on the other side of the sheet.

The cog railroad that runs to the summit of Pikes Peak makes, at the steepest place, a 30° angle with the horizontal. How many meters would you rise in going 400 meters along the track at this part of the road?



In a 30-60-90 triangle, the side opposite the 300 angle is half the length of the hypotenuse, so h = 200 m

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SCHOOL: _____

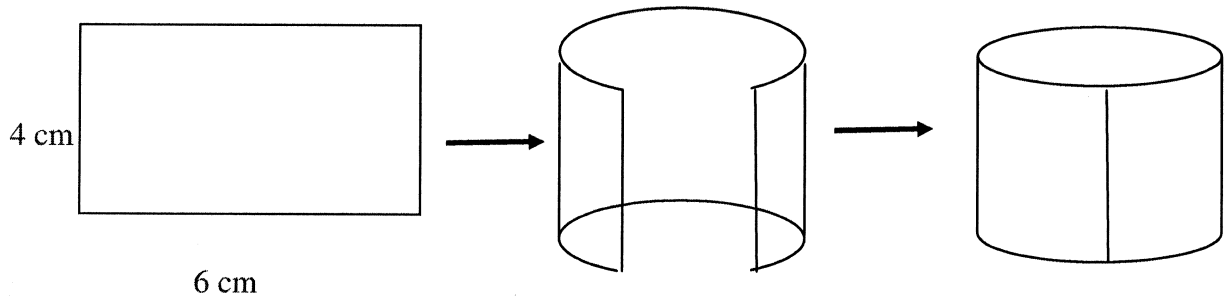
PROBLEM 7 **TIME: 4 minutes**

$9/\pi$

answer

Put no work on this side of the paper. Write the answer and only the answer in the space above. Put all work on the other side of the sheet.

The figure shows how a rectangular piece of paper is rolled to form a cylindrical tube. If it is assumed that the 4-centimeter sides of the rectangle meet with no overlap, what is the area, in square centimeters, of the base of the cylindrical tube?



The circumference of the base is 6.

$C = 2\pi r$

$6 = 2\pi r$

$r = \frac{6}{2\pi} = \frac{3}{\pi}$

$A = \pi r^2$

$= \pi \left(\frac{3}{\pi}\right)^2 = \pi \left(\frac{9}{\pi^2}\right) = \frac{9}{\pi}$

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TEAM: _____

SCHOOL: _____

PROBLEM 8 **TIME: 3 minutes**

$$\frac{1}{1000}$$

answer

Put no work on this side of the paper. Write the answer and only the answer in the space above. Put all work on the other side of the sheet.

Compute and write as a reduced fraction: $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{1000}\right)$

$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdots\frac{998}{999}\cdot\frac{999}{1000}$

All the numerators cancel except 1.

All the denominators cancel ~~except~~ except 1000.

$\frac{1}{1000}$

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NAME: SOLUTION

TEAM: _____

SCHOOL: _____

PROBLEM 9 **TIME: 5 minutes**

6

answer

Put no work on this side of the paper. Write the answer and only the answer in the space above. Put all work on the other side of the sheet.

How many of the factors of 2! 3! 5! are perfect squares?

$$2!\ 3!\ 5! = 2 \cdot 3 \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 2$$

$$= 2 \cdot 3 \cdot 2 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2$$

$$= 2^5 \cdot 3^2 \cdot 5$$

All perfect squares must be a product of an even number of 2's and ~~an~~ an even number of 3's.

There are three ways to make an even number of 2's: $(2^0, 2^2, 2^4)$ and two ways to make an even number of 3's: $(3^0, 3^2)$

So there are 6 such factors.

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NAME: _____

TEAM: _____

SCHOOL: _____

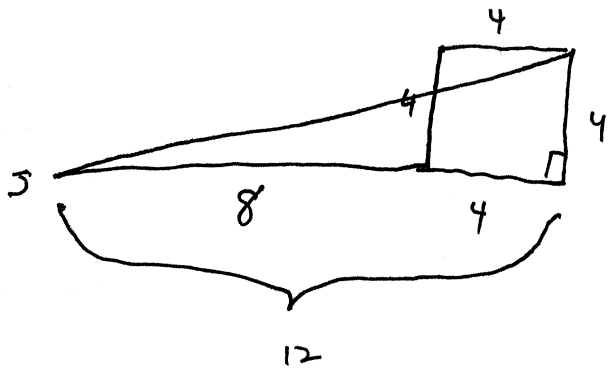
PROBLEM 10 **TIME: 4 minutes**

$4\sqrt{10}$ mi

answer

Put no work on this side of the paper. Write the answer and only the answer in the space above. Put all work on the other side of the sheet.

Two hikers left from the same place at the same time. Both hiked at a constant speed of 4 miles per hour. Josh hiked due west for 2 hours, while Mandy hiked due north for one hour and then due east for the second hour. How far apart were the hikers, in miles, at the end of the two hours?



The distance s must be given by

$$s^2 = 12^2 + 4^2$$

$$s^2 = 4^2 \cdot 3^2 + 4^2$$

$$s^2 = 4^2(3^2 + 1)$$

$$s^2 = 4^2 \cdot 10$$

$$s = 4\sqrt{10}$$

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Saturday 17 March 2012

TEAM: SOLUTION

SCHOOL: _____

TEAM PROBLEMS **TIME: 45 minutes**

1. 2ℓ
2. $(-\infty, -5] \cup [-2, 2]$
3. $30 + 10\sqrt{3}$ ft
4. 72
5. 215, 287, 407, 527, 551

Put no work on this side of the paper. Write the answers only in the above spaces. Put all work on the enclosed sheets of scrap paper, and hand in the scrap paper with your answer sheet.

1. A car radiator is filled with 5 liters of a 25% antifreeze solution. How many liters must be drawn off and replaced by a 75% antifreeze solution to leave the radiator filled with a 55% antifreeze solution?

2. Solve for x:
$x^3 + 5x^2 - 4x - 20 \leq 0$

3. Two buildings are separated by an alley. Joe is looking out a window 30 feet above the ground in one of the buildings. He observes the measurement of the angle of depression of the base of the second building to be 60° and the angle of elevation of the top of the second building to be 45°. How tall is the second building in feet?

4. How many factors does 17640 have?

5. The two roots of the quadratic equation $x^2 - 48x + c = 0$ are both prime numbers. Find all possible positive values of c.

1. Building the equation around antifreeze:

$$\underbrace{(0.25)5}_{\text{orig. amt}} + \underbrace{0.75x}_{\text{added}} = \underbrace{(0.55)5}_{\text{final}}$$

$$1.25 + 0.75x = 2.75$$

$$0.75x = 1.5$$

$$x = 2\ \ell$$

2. $$x^3 + 5x^2 - 4x - 20 \leq 0$$

$$x^2(x+5) - 4(x+5) \leq 0$$

$$(x^2-4)(x+5) \leq 0$$

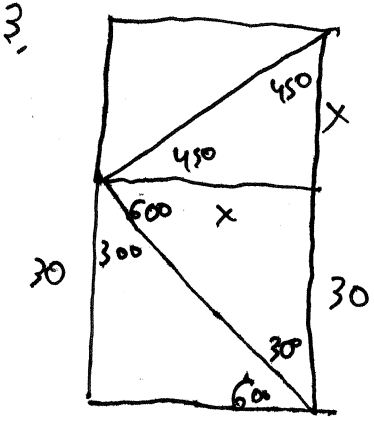
$$(x+5)(x+2)(x-2) \leq 0$$

Zeroes occur at $-5, -2, 2$.

Obviously $x = 0$ does work, and working and non-working regions alternate. And the endpoints work.

$$(-\infty, -5] \cup [-2, 2]$$

3.



$\tan 30° = \frac{x}{30}$

$x = 30 \tan 30°$

$x = 30 \frac{\sqrt{3}}{3} = 10\sqrt{3}$

So the height is $30 + 10\sqrt{3}$ ft

4. $17640 = 10 \cdot 1764 = 2 \cdot 5 \cdot 9 \cdot 196 = 2 \cdot 5 \cdot 3^2 \cdot 14^2 = 2 \cdot 5 \cdot 3^2 \cdot 2^2 \cdot 7^2$

$= 2^3 \cdot 3^2 \cdot 5 \cdot 7^2$

4 possibilities of factors of 2

3 possibilities of factors of 3

2 possibilities of factors of 5

3 possibilities of factors of 7

$4 \cdot 3 \cdot 2 \cdot 3 = 72$

5. sum of roots is $-b/a = 48$. Which primes sum to 48?

$48 = 3 + 45$

$= \boxed{5 + 43}$

$= \boxed{7 + 41}$

$= \boxed{11 + 37}$

$= 13 + 35$

$= \boxed{17 + 31}$

$= \boxed{19 + 29}$

$= 23 + 25$

Product of roots is $c/a = c$.

Possible products: $5 \cdot 43 = 215$

$7 \cdot 41 = 287$

$11 \cdot 37 = 407$

$17 \cdot 31 = 527$

$19 \cdot 29 = 551$

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SCHOOL: SOLUTION

TEAM: _____

RELAY: 1

1. 60
2. 3
3. 7
4. 21

Third Annual Upper Peninsula
High School Math Challenge
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(MARQUETTE, MI, USA)
Saturday 17 March 2012

SCHOOL: SOLUTION

TEAM: _____

RELAY: 2

1. 7
2. $\frac{7\sqrt{3}}{2}$
3. 25
4. 9

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High School Math Challenge
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Saturday 17 March 2012

SCHOOL: SOLUTION

TEAM: _____

RELAY: 3

1. -3
2. 1
3. $\frac{10}{21}$
4. 3

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Saturday 17 March 2012

SCHOOL: _____

TEAM: _____

RELAY: _____

1. _____
2. _____
3. _____
4. _____

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RELAY 1 **PLAYER 1**

Category: LEAST COMMON MULTIPLES

What is the least common multiple of 1, 2, 3, 4, 5, and 6?

Pass your answer to Player 2.

$1, 2, 3, 2^2, 5, 2 \cdot 3$

LCM is $2^2 \cdot 3 \cdot 5 = 60$

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RELAY 1 **PLAYER 2**

Category: WORK PROBLEM

My sister and I, working alone, can each shovel the driveway in five hours. The number you will receive from Player 1 is the number of minutes I work alone before my sister joins me. After my sister joins me, we work together to finish shoveling the driveway. How many hours will I be shoveling?

$60_{min} = 1\ hr$

	r	t	d
me	$\frac{1}{5}$	x	$\frac{1}{5}x$
sister	$\frac{1}{5}$	$x-1$	$\frac{1}{5}(x-1)$

$$\frac{1}{5}x + \frac{1}{5}(x-1) = 1$$

$$x + x - 1 = 5$$

$$2x - 1 = 5$$

$$2x = 6$$

$$x = 3$$

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RELAY 1 **PLAYER 3**

Category: GEOMETRIC SERIES

The number you will receive is the number of pies I have for sale. If the cost of the first pie is one dollar and each subsequent pie costs twice as much as the one before it, how many dollars will I get from selling my pies?

Pass your answer to Player 4.

3 pies

$1 + $2 + $4 = $7

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RELAY 1 **PLAYER 4**

Category: PERMUTATIONS AND COMBINATIONS

Let n be the number you will receive from Player 3. I am holding n cards in my hand. If two of the cards are identical to each other, and the other $n-2$ cards are identical to each other, in how many ways can I arrange the cards in my hand?

Run your answer to the front.

7 cards

$$\text{arrangements} = \frac{7!}{2!\,5!} = \frac{7 \cdot 6 \cdot 5!}{2!\,5!} = \frac{7 \cdot 6}{2} = 21$$

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RELAY 2 **PLAYER 1**

Category: EQUATIONS OF LINES

Points (–2, 3), (2, 5), and (6, k) are collinear. Find k.

Pass your answer to Player 2.

slopes must be the same

$$\frac{5-3}{2--2} = \frac{k-5}{6-2}$$

$$\frac{2}{4} = \frac{k-5}{4}$$

$$k-5=2$$

$$k=7$$

Third Annual Upper Peninsula High School Math Challenge

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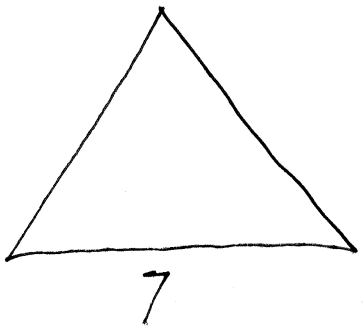
RELAY 2 **PLAYER 2**

Category: GEOMETRY

The number you will receive from Player 1 is the length of a side of an equilateral triangle. What is the height of this triangle?

Pass your answer to Player 3.



$$h = \frac{s\sqrt{3}}{2}$$

$$= \frac{7\sqrt{3}}{2}$$

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RELAY 2 **PLAYER 3**

Category: INTEGER FACTORING

Let n be the number you will receive from Player 2.

Express n^2 as a reduced improper fraction. Subtract the denominator from the numerator of this fraction. Call this number m.

Compute the sum of all proper factors (all factors other than the number itself) of m.

Pass your answer to Player 4.

$n = \frac{7\sqrt{3}}{2}$

$n^2 = \frac{49 \cdot 3}{4} = \frac{147}{4}$ $\quad m = 143$

Factors are $1, 11, 13, 143$

$1 + 11 + 13 = 25$

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RELAY 2 **PLAYER 4**

Category: MIXTURE PROBLEM

The number you will receive from Player 3 is the value of the money (in U.S. dollars) that I have in my pocket. I only have fives and singles. If I have one more single than I have five dollar bills, how many bills do I have?

Run your answer to the front.

I have $25

I have x singles and x-1 fives

$x + 5(x-1) = 25$

$x + 5x - 5 = 25$

$6x - 5 = 25$

$6x = 30$

$x = 5$

5 singles 4 fives

9 bills

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RELAY 3 **PLAYER 1**

Category: SUM AND PRODUCT OF ROOTS

The equation $x^3 + 2x^2 - 13x + 10 = 0$ has three real roots. One of the roots is 1. What is the sum of the other two roots?

Pass your answer to Player 2.

Sum of roots $= -\frac{b}{a} = -2$

$r_1 + r_2 + 1 = -2$

$r_1 + r_2 = -3$

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RELAY 3 **PLAYER 2**

Category: ARITHMETIC SERIES

The number, d, that you will receive from Player 1 is the common difference between terms in an arithmetic series. $[(a_{i+1} - a_i) = d]$. If the first term in the series is 10, what is the fourth term?

Pass your answer to Player 3.

$d = -3$

series is 10, 7, 4, 1

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RELAY 3 **PLAYER 3**

Category: PROBABILITY

The number you will receive from Player 2 is n. In a bin there are $5n$ red balls and $n + 1$ blue balls. If two balls randomly fall out of the bin, what is the probability that they are of different colors?

Pass your answer to Player 4.

$n = 1$

so 5 red balls and 2 blue balls

There are ${}_7C_2$ pairs: $\frac{7!}{2!\,5!} = \frac{7 \cdot 6}{2} = 21$

10 ways (5·2) to grab balls of different colors

$\frac{10}{21}$

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RELAY 3 **PLAYER 4**

Category: PRIME NUMBERS

The number you will receive from Player 3 is a fraction. Make sure it is in reduced form.

Let n be the sum of the numerator and denominator of that fraction.

Divide this number by the largest prime palindrome with an even number of digits.
(A palindrome is a number that reads the same forwards as backwards. Example: 131)

Round your answer up to the nearest integer. This is your answer.

Run your answer to the front.

$\frac{10}{21}$

$n = 33$

The only prime palindrome with an even number of digits is 11, since all even-length palindromes are divisible by 11.

$\frac{33}{11} = 3$