Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Team:____

School:____

- ☐ Please don't put any work on this page, use the scrap paper provided.
- ☐ Write only the answer to each problem in the spaces below.
- ☐ When you are finished, return all papers in the unsealed envelope.

Team Problems **Time** (45 min)

1. ____________________________________
2. ____________________________________
3. ____________________________________
4. ____________________________________
5. ____________________________________

1. Consider the recusive sequence of numbers $\{u_1, u_2, u_3, ...\}$ given by:

$$u_1 = 3, \qquad u_n = \frac{-1}{1 + u_{n-1}}, \qquad \text{for } n > 1.$$

Find the 2019-th term in the sequence, u_{2019}.

Solution. We have that:

$$\begin{aligned} u_1 &= 3 \\ u_2 &= \frac{-1}{1+3} = -\frac{1}{4} \\ u_3 &= \frac{-1}{1+\left(-\frac{1}{4}\right)} = -\frac{4}{3} \\ u_4 &= \frac{-1}{1+\left(-\frac{4}{3}\right)} = 3. \end{aligned}$$

This will imply that any $u_{3k+1} = 3$, $u_{3k+2} = -\frac{1}{4}$, and $u_{3k} = -\frac{4}{3}$. Since 2019 is a multiple of 3 ($2019 = 1800 + 210 + 9 = 3(673)$), we must have that $u_{2019} = -\frac{4}{3}$.

2. Find the area of the region bounded by the graphs of the three lines: $y = 3x$, $x = 3y$, and $3x + y = 30$.

Solution. The first and second lines intersect at $A = (0, 0)$. The second and third lines intersect when:

$$\begin{aligned} \frac{x}{3} &= 30 - 3x \\ 10x &= 90 \\ x &= 9, \end{aligned}$$

and so they intersect at $B = (9, 3)$. The first and third lines intersect when:

$$\begin{aligned} 3x &= 30 - 3x \\ 6x &= 30 \\ x &= 5, \end{aligned}$$

and so they intersect at $C = (5, 15)$. Notice that the triangle ABC is a right-triangle since the second and third lines have negative-reciprocal slopes. The area can thus be computed:

$$\text{area}(ABC) = \frac{|AB| \cdot |BC|}{2} = \frac{3\sqrt{10} \cdot 4\sqrt{10}}{2} = 60 \text{ units}^2.$$

3. Find three **consecutive** odd numbers with the following property:

the sum of the squares of the three consecutive odd numbers is a four digit number with all four digits the same.

Some examples of three consecutive odd numbers include: 15, 17, 19 or 23, 25, 27.

Solution. The three consecutive odd numbers can be written in the form $\{2k - 1, 2k + 1, 2k + 3\}$, for some integer k. Consider writing the sum of their squares in the following way:

$$\begin{aligned} (2k-1)^2 + (2k+1)^2 + (2k+3)^2 &= 4k^2 - 4k + 1 + 4k^2 + 4k + 1 + 4k^2 + 12k + 9 \\ &= 12k^2 + 12k + 11. \end{aligned}$$

Notice that since the sum of their squares is an odd number with all four digits the same, it must be 1111, 3333, 5555, 7777, or 9999.

In the first case we have:

$$\begin{aligned} 12k^2+12k+11 &= 1111 \\ 12k(k+1) &= 1100 \\ 2^2\cdot 3\cdot k(k+1) &= 2\cdot 2\cdot 5^2\cdot 11 \end{aligned}$$

but this is impossible since the right hand side isn't divisible by 3. Similarly,

$$\begin{aligned} 12k^2+12k+11 &= 3333 \\ 12k(k+1) &= 3322 \\ 2^2\cdot 3\cdot k(k+1) &= 2\cdot 1661 \end{aligned}$$

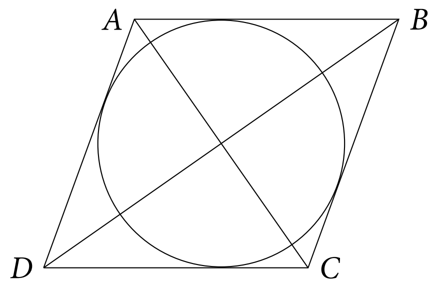
which is also impossible since the right hand side isn't divisible by 3 (the sum of digits of 1661 isn't divisible by 3, for example).

The next case is:

$$\begin{aligned} 12k^2+12k+11 &= 5555 \\ 12k(k+1) &= 5544 \\ 2^2\cdot 3\cdot k(k+1) &= 2\cdot 2772 \\ 2^2\cdot 3\cdot k(k+1) &= 2^2\cdot 1386 \\ 2^2\cdot 3\cdot k(k+1) &= 2^2\cdot 3\cdot 462 \\ 2^2\cdot 3\cdot k(k+1) &= 2^3\cdot 3\cdot 231 \\ 2^2\cdot 3\cdot k(k+1) &= 2^3\cdot 3^2\cdot 77 \\ 2^2\cdot 3\cdot k(k+1) &= 2^3\cdot 3^2\cdot 7\cdot 11 \end{aligned}$$

which DOES have a solution $2^2\cdot 3\cdot k(k+1) = 2^2\cdot 3\cdot 21\cdot 22$. Therefore $k = 21$, and so the three consecutive odd numbers are: $\{2(21)-1, 2(21)+1, 2(21)+3\} = \{41, 43, 45\}$.

4. Consider the rhombus ($AB = BC = CD = AD$) shown below, with diagonals $AC = 12$ cm and $BD = 16$ cm. Find the exact diameter of the inscribed circle.



Solution. Let the radius of the circle be r and the center of the circle be O. Let the top point of tangency of the circle with the rhombus be E. We have that $AO = 6$, $OB = 8$, and so by the Pythagorean Theorem, $AB = 10$. Let $EB = x$, and therefore $AE = 10 - x$.

The triangles AOB, OEB, and OEA are similar, and so we have the equality of ratios:

$$\frac{6}{8} = \frac{r}{x} \qquad AND \qquad \frac{6}{8} = \frac{10-x}{r}$$

$$\Updownarrow$$

$$3x = 4r \qquad AND \qquad 3r = 40 - 4x.$$

Solving this equations gives:

$$\begin{aligned} 3r &= 40 - 4\left(\frac{4r}{3}\right) \\ 9r &= 120 - 16r \\ 25r &= 120 \\ r &= \frac{120}{5} = \frac{24}{5}. \end{aligned}$$

Thus the diameter of the inscribed circle is $\dfrac{48}{5} = 9\tfrac{3}{5} = 9.6$ cm.

5. How many zeros are at the end of the following number?

$$2019! = 2019 \times 2018 \times 2017 \times \cdots \times 3 \times 2 \times 1$$

Solution. One can see that the number 2019! will be divisible more times by 2 than by 5. So we just need to figure out how many times it is divisible by 5 (call that number b) then $2019! = A \cdot 2^a \cdot 5^b$, where $a > b$, and $2019! = A \cdot 2^{a-b} \cdot (10)^b$, implying that it ends in b many zeros.

To count the number of times 2019! is divisible by 5, we can **add** the number of multiples of 5, 25, 125, and 625 that lie within the range $\{1, \ldots, 2019\}$. There are 403 multiples of 5 in that range, 80 multiples of 25, 16 multiples of 125, and 3 multiples of 625.

Therefore the highest power of 5 that divides 2019! is $5^{403+80+16+3} = 5^{502}$, which by our earlier argument, implies that 2019! ends in 502 zeros.