Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Relay 1 | Category: Prime Numbers | Player 1

What is the 15th prime number?

Pass your answer to Player 2.

Solution. To check that a number is prime, one only needs to check non-divisibility of primes less than or equal to its square root, so for example, one can convince themselves that 43 is prime by showing that it is not divisible by 2 (its not even), nor 3 (the sum of its digits is not a multiple of 3), nor 5 (it doesn't end in 5 or 0).

One compiles the list: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. So the 15th prime number is 47.

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Relay 1 | Category: Quadratic Equations | Player 2

Let n be the number you receive from PLAYER 1.

Consider the equation $x^2 - nx + k = 0$. You don't know what k is, but you do know that -1 is a root of this equation. What is the other root?

Pass your answer to PLAYER 3.

Solution. The number from player 1 is $n = 47$. Let the unknown root be r. The sum of the two roots of a quadratic $x^2 + bx + c$ is always equal to $-b$. Therefore, $-1 + r = 47$. This gives that $r = 48$.

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RELAY 1 | Category: PERFECT SQUARES | PLAYER 3

Let n be the number that you receive from Player 2.

If a and b are positive consecutive odd integers and $a^2 - b^2 = n$, what is b?

Pass your answer to Player 4.

Solution. The number from player 2 is $n = 48$. Since a and b are consecutive odd integers, $a - b = 2$. We have:

$$\begin{aligned} a^2 - b^2 &= 48 \\ (a+b)(a-b) &= 48 \\ (a+b) \cdot 2 &= 48 \\ a + b &= 24 \end{aligned}$$

Therefore $a = 13$ and $b = 11$.

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Relay 1 | Category: Palindromes | Player 4

Let n be the number you receive from Player 3.

What is the smallest whole number k for which n^k is **not** a palindrome (in base ten)?

Run your answer to the front.

Solution. The number from player 3 is $n = 11$.

- $11^1 = 11$ is a palindrome (a number that reads the same in both directions)
- $11^2 = 121$ is a palindrome
- $11^3 = 121 \cdot (10 + 1) = 1210 + 121 = 1331$ is a palindrome
- $11^4 = 1331 \cdot (10 + 1) = 13310 + 1331 = 14641$ is a palindrome
- $11^5 = 14641 \cdot (10 + 1) = 146410 + 14641 = 161051$ is not a palindrome

Therefore $k = 5$.

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Relay 2 | Category: Venn Diagrams | Player 1

Among a group of students,

15 like English, 5 like Math, and 10 like History.

4 students like English and Math, 8 like English and History, and 2 like Math and History.

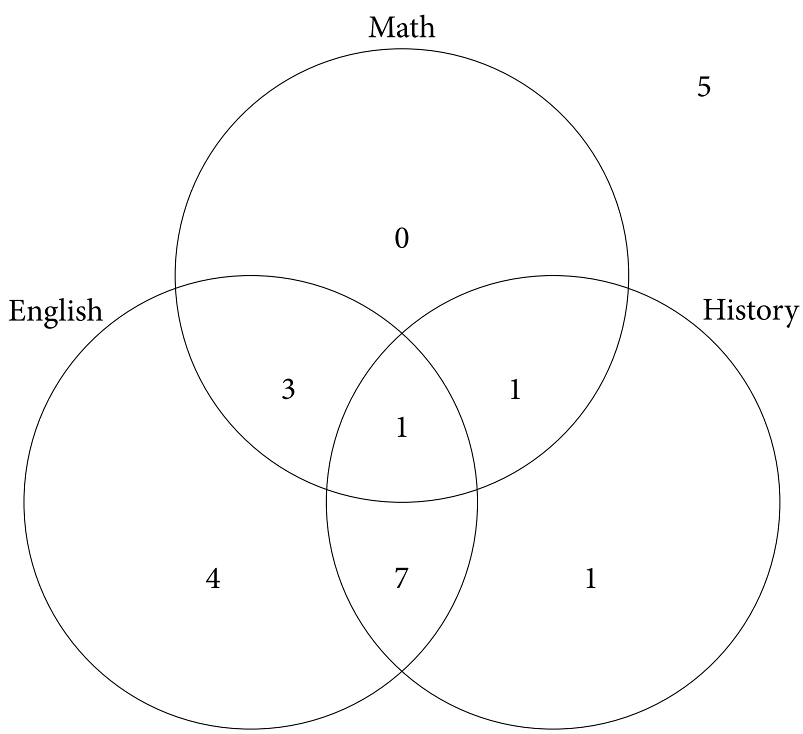
1 student likes all three subjects.

5 students don't like any of the three subjects.

How many students are in this group?

Pass your answer to PLAYER 2.

Solution.



There are 22 students in the group.

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Relay 2 | Category: Approximations to π | Player 2

Let n be the number you receive from Player 1.

What is $\frac{n}{\pi}$ rounded to the nearest integer?

Pass your answer to Player 3.

Solution. The number from player 1 is $n = 22$. The approximation $\pi \approx \frac{22}{7}$ is accurate to two decimal places. Therefore $\frac{22}{7} - \frac{1}{100} < \pi < \frac{22}{7} + \frac{1}{100}$. This means:

$$\frac{22}{\frac{22}{7} + \frac{1}{100}} < \frac{22}{\pi} < \frac{22}{\frac{22}{7} - \frac{1}{100}}$$

$$7 - \frac{7}{2207} < \frac{22}{\pi} < 7 + \frac{7}{2193}.$$

Therefore $\frac{22}{7}$ rounded to the nearest integer is 7.

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Relay 2 | Category: Remainders | Player 3

Let n be the number you receive from Player 2.

A *googol* is the number 1 followed by one hundred zeros. If a googol is divided by n, what is the remainder?

Pass your answer to Player 4.

Solution. The number from player 2 is $n = 7$. A googol is 10^{100}. Take a look at the sequence of remainders when powers of ten are divided by 7.

- 10^1 has remainder 3.
- 10^2 has remainder 2
- 10^3 has remainder 6
- 10^4 has remainder 4
- 10^5 has remainder 5
- 10^6 has remainder 1
- 10^7 has remainder 3

This sequence repeats every 6 iterations. Since $100 = 6(16) + 4$, the remainder when 10^{100} is divided by 7 is the same as when 10^4 is divided by 7. The remainder is therefore 4.

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Relay 2 | Category: Coins | Player 4

Let n be the number you receive from Player 3.

I have quarters and dimes on the table, 10 coins in all. I have n more quarters than dimes.

What is my total amount of money, in cents?

Run your answer to the front.

Solution. The number from player 3 is $n = 4$.

Let q be the number of quarters and d be the number of dimes. We have that $q = d + 4$ and $q + d = 10$. These equations give $(d + 4) + d = 10$, $2d = 6$, $d = 3$. So there are 3 dimes and 7 quarters. The total money (in cents!) is thus $25(7) + 10(3) = 205$.

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Relay 3 | Category: Infinite Series | Player 1

Consider the infinite series $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$

Write the sum of this series in the form $\frac{p}{q}$ in lowest terms. What is $p + q$?

Pass your answer to Player 2.

Solution. We have a geometric series $1 + r + r^2 + r^3 + \ldots$ with common ratio $r = \frac{2}{3}$. The infinite sum can be obtained:

$$1 + r + r^2 + \ldots = \frac{1}{1 - r} = \frac{1}{1 - \frac{2}{3}} = 3 = \frac{3}{1}.$$

Therefore $\frac{p}{q} = \frac{3}{1}$ and so $p + q = 3 + 1 = 4$.

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Relay 3	Category: Circles and Squares	Player 2

Let n be the number you receive from PLAYER 1.

A square of side length n is inscribed in a circle. If the area of the circle is $k\pi$, what is k?

Pass your answer to PLAYER 3.

Solution. The number from player 1 is $n = 4$. A square of side length 4 has a diagonal length of l where $l^2 = 4^2 + 4^2$, and so $l = 4\sqrt{2}$. The radius of the circle is therefore $\frac{l}{2} = 2\sqrt{2}$, and so the area of the circle must be $\pi(2\sqrt{2})^2 = 8\pi$. Thus $k = 8$.

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Relay 3 | Category: Hamburger Problem | Player 3

Let n be the number that you receive from Player 2.

Andy can eat three times as many hamburgers as Dan. If together they ate n hamburgers, how many hamburgers did Andy eat?

Pass your answer to Player 4.

Solution. The number from player 2 is $n = 8$. Let A, D be the number of hamburgers they eat, respectively. Then $8 = A + D = 3D + D$, $8 = 4D$, $2 = D$, $6 = A$.

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Relay 3 Category: Cubic Equations Player 4

Let n be the number you receive from PLAYER 3.

Consider the equation: $x^3 - nx^2 + 11x - n = 0$.

What is the sum of the largest real root and the smallest real root of this equation?

Run your answer to the front.

Solution. The number from player 3 is $n = 6$. Consider the cubic equation $x^3 - 6x^2 + 11x - 6 = 0$. Let the three roots be $\{r_1, r_2, r_3\}$. We know that the sum of the three roots must be the negative of the coefficient of x^2, and the product of the three roots must be the negative of the constant coefficient. Therefore:

$$\begin{aligned} r_1 + r_2 + r_3 &= 6 \\ r_1 r_2 r_3 &= 6. \end{aligned}$$

We can solve these equations with $\{r_1, r_2, r_3\} = \{1, 2, 3\}$. Therefore the answer is $1 + 3 = 4$.