Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Name:_____

Team:_____

School:_____

Problem 1 (Lines and Slopes) **Time** (3 min)

Don't turn over this page until instructed. Please don't work on this side of the paper.
Write only the answer in the box.

Suppose the points $(m, 3)$ and $(1, m)$ lie on a line of slope m, where m is positive.

Find the value of m.

Solution. Let $(x_1, y_1) = (m, 3)$ and $(x_2, y_2) = (1, m)$. The slope can be computed:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{m - 3}{1 - m},$$

which yields:

$$\begin{aligned} m(1 - m) &= m - 3 \\ m - m^2 &= m - 3 \\ m^2 &= 3 \\ m &= \sqrt{3}. \end{aligned}$$

Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Name:_____

Team:_____

School:_____

Problem 2 (Averages) **Time** (4 min)

Don't turn over this page until instructed. Please don't work on this side of the paper.
Write only the answer in the box.

One class with M students got an average of 90% on a test. Another class with N students got an average of 80% on the same test.

The combined average of the two classes is 82%.

What is the value of $\dfrac{M}{N}$?

Solution. The total accumulated points from the two classes combined, can be written in two different ways:

$$\begin{aligned} 90M + 80N &= 82(M+N) \\ 8M &= 2N \\ \frac{M}{N} &= \frac{2}{8} = \frac{1}{4}. \end{aligned}$$

Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

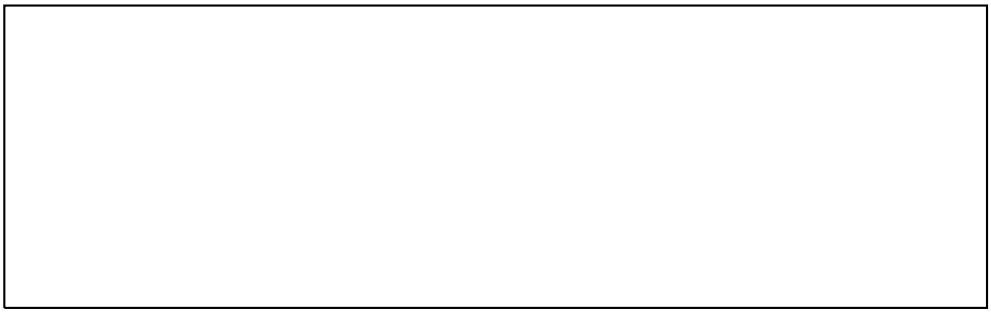
Name:_____

Team:_____

School:_____

Problem 3 (Reflections and Triangles) **Time** (4 min)

Don't turn over this page until instructed. Please don't work on this side of the paper.
Write only the answer in the box.



The point $A = (2, 3)$ is reflected about the x-axis to another point B. The point B is reflected about the line $y = x$ to a third point C.

Find the area of the triangle ABC.

Solution. $A = (2, 3)$, $B = (2, -3)$, $C = (-3, 2)$. To compute the area:

$$\text{area} = \frac{\text{base} \cdot \text{height}}{2} = \frac{5 \cdot 6}{2} = 15 \text{ units}^2.$$

Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Name:_____

Team:_____

School:_____

Problem 4 (Ratios) **Time** (3 min)

Don't turn over this page until instructed. Please don't work on this side of the paper.
Write only the answer in the box.

The ratio of a to b is $4 : 3$.

The ratio of c to d is $2 : 1$.

The ratio of d to b is $1 : 6$.

What is the ratio of a to c?

Solution. We have that $\frac{a}{b} = \frac{4}{3}$, $\frac{c}{d} = \frac{2}{1}$, and $\frac{d}{b} = \frac{1}{6}$.

Therefore $\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{d} \cdot \frac{d}{c} = \frac{4}{3} \cdot \frac{6}{1} \cdot \frac{1}{2} = \frac{4}{1}$.

This gives that the ratio of a to c is $4 : 1$.

Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Name:_____

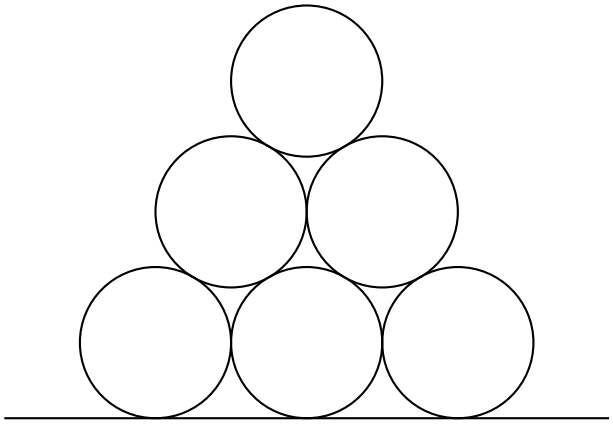
Team:_____

School:_____

Problem 5 (Geometry) **Time** (4 min)

Don't turn over this page until instructed. Please don't work on this side of the paper.
Write only the answer in the box.

Six circles, each with a two foot diameter, are stacked tangent to each other as shown.



What is the height of the stack?

Solution. Draw an equilateral triangle between the three centers of the extreme outer circles. It will have a side length of 4. Let h be its height.

By the Pythagorean Theorem we have:

$$\begin{aligned} 4^2 &= h^2 + 2^2 \\ h^2 &= 12 \\ h &= \sqrt{12} \\ h &= 2\sqrt{3}. \end{aligned}$$

Therefore the height of the stack is $1 + \sqrt{12} + 1 = 2 + \sqrt{12} = 2 + 2\sqrt{3}$ feet.

Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Name:_____

Team:_____

School:_____

Problem 6 (Solving Equations) **Time** (4 min)

Don't turn over this page until instructed. Please don't work on this side of the paper.
Write only the answer in the box.

Consider the equation:

$$\frac{2y}{x} = \frac{y-2x}{x-3y}$$

for x and y positive numbers.

Find the value of $\dfrac{x}{y}$.

Solution. Let $a = \frac{x}{y}$. The equation can be rewritten:

$$\begin{aligned}
2\left(\frac{y}{x}\right) &= \frac{\left(\frac{1}{x}\right)(y-2x)}{\left(\frac{1}{x}\right)(x-3y)} \\
2\left(\frac{y}{x}\right) &= \frac{\left(\frac{y}{x}\right)-2}{1-3\left(\frac{y}{x}\right)} \\
2\left(\frac{1}{a}\right) &= \frac{\left(\frac{1}{a}\right)-2}{1-3\left(\frac{1}{a}\right)} \\
\frac{2}{a} &= \frac{1-2a}{a-3} \\
2(a-3) &= a(1-2a) \\
2a^2+a-6 &= 0 \\
2a^2+4a-3a-6 &= 0 \\
2a(a+2)-3(a+2) &= 0 \\
(a+2)(2a-3) &= 0.
\end{aligned}$$

We want the positive solution $a = \frac{x}{y} = \frac{3}{2}$.

Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Name:_____

Team:_____

School:_____

Problem 7 (Sum of a Sequence) **Time** (4 min)

Don't turn over this page until instructed. Please don't work on this side of the paper.
Write only the answer in the box.

What is the sum of the first 100 odd numbers?

Solution. Adding up consecutive odd numbers always gives a perfect square.

$$\begin{aligned} 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2 \\ &\vdots \\ 1+3+5+\cdots+(2n-1) &= n^2. \end{aligned}$$

Therefore $1+3+5+7+\cdots+197+199 = 100^2 = 10,000$.

Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Name:_____

Team:_____

School:_____

Problem 8 (Exponents and Digits) **Time** (4 min)

Don't turn over this page until instructed. Please don't work on this side of the paper.
Write only the answer in the box.

How many zeros are at the end of the number $20^{50} \times 50^{20}$?

Solution.

$$\begin{aligned} 20^{50} \cdot 50^{20} &= (2^2 \cdot 5)^{50} \cdot (2 \cdot 5^2)^{20} \\ &= 2^{100} \cdot 5^{50} \cdot 2^{20} \cdot 5^{40} \\ &= 2^{120} \cdot 5^{90} \\ &= 2^{30} \cdot 2^{90} \cdot 5^{90} \\ &= 2^{30} \cdot (2 \cdot 5)^{90} \\ &= 2^{30} \cdot 10^{90}. \end{aligned}$$

Therefore the number ends in 90 zeros.

Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Name:_____

Team:_____

School:_____

Problem 9 (More Sums of Sequences) **Time** (4 min)

Don't turn over this page until instructed. Please don't work on this side of the paper.
Write only the answer in the box.

What is the numerical value of the expression?

$$100^2 - 99^2 + 98^2 - 97^2 + \cdots + 2^2 - 1^2$$

Solution. Let the expression be E.

$$\begin{aligned}
E &= (100-99)(100+99) + (98-97)(98+97) + \cdots + (4-3)(4+3) + (2-1)(2+1) \\
&= 199 + 195 + 191 + \cdots + 7 + 3 \\
&= (4(49)+3) + (4(48)+3) + \cdots + (4(2)+3) + (4+3) + 3 \\
&= 4(49+48+\cdots+2+1) + 50(3) \\
&= 4\left(\frac{49\cdot 50}{2}\right) + 50(3) \\
&= 51(50) \\
&= 5,050.
\end{aligned}$$

Tenth Annual Upper Peninsula High School Math Challenge

Northern Michigan University

4/6/19

Name:_____

Team:_____

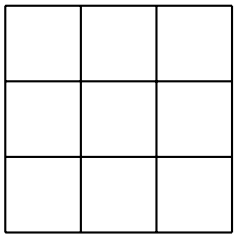
School:_____

Problem 10 (Counting and Probability) **Time** (5 min)

Don't turn over this page until instructed. Please don't work on this side of the paper.
Write only the answer in the box.

Three darts land in **three different** squares when thrown at the board below.

What is the probability that the three darts land in squares that form a horizontal, vertical, or diagonal line?



Solution. Since the three darts land in three different squares, there are

$$9C3 = \binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 3 \cdot 4 \cdot 7 = 84$$

possible ways this could happen. Of these 84 possibilities, there are only 8 lines (3 vertical, 3 horizontal, 2 diagonal). Therefore the probability is:

$$\frac{8}{84} = \frac{4}{42} = \frac{2}{21}.$$