First Annual Upper Peninsula High School Math Challenge

NORTHERN MICHIGAN UNIVERSITY

Saturday 13 March 2010

Solutions Page 1

INDIVIDUAL PROBLEMS

1. Ten boys and ten girls showed up for the sixth grade dance. At some point, each boy danced with each girl. After about an hour of dancing, the boys removed themselves to the corner to discuss football, but the girls continued dancing, and each girl danced with each other girl at some point.

Counting all the boy-girl dancing pairs and all the girl-girl dancing pairs, how many **TOTAL** dancing pairs were there?

ANSWER: 145

Solution: With 10 boys and 10 girls to choose from, there must be (10)(10), or 100 boy-girl pairs. When the girls dance with each other, all 10 girls dance with 9 other girls, which would give 90 combinations, except that each girl-girl pair is counted twice. There must be 45 girl-girl pairs, for a grand total of 145.

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2. That darn ditto machine! I've been given a polynomial to factor and it looks like this:

$6x^2 - 11x$ <<<BLOB>>>

I don't know what that blob was supposed to be, but it's a fair guess that it represents the *constant* term in the polynomial. Due to my psychic powers, I happen to know that one of the factors is $2x + 3$. What is the **other factor**?

ANSWER: $3x - 10$

Solution: Using polynomial division:

```
            3x   - 10_____
2x + 3 |  6x^2 - 11x + B
          6x^2 +  9x
               - 20x + B
               - 20x -30
                  B + 30
```

Since this division is supposed to come out even, the other factor must be $3x - 10$, and the remainder is 0, implying that the blob must be -30.

Alternatively, using FOIL, $6x^2 - 11x + B = (2x+3)(ax+b) = 2ax^2 + (2b+3a)x + 3b$.

So, matching coefficients, $2a = 6$; $2b+3a = –11$; $3b = B$. We know from the first equation that $a = 3$. Substituting into the second, we get $2b+9 = –11$, so $2b = –20$, so $b = –10$, giving us $3x – 10$.

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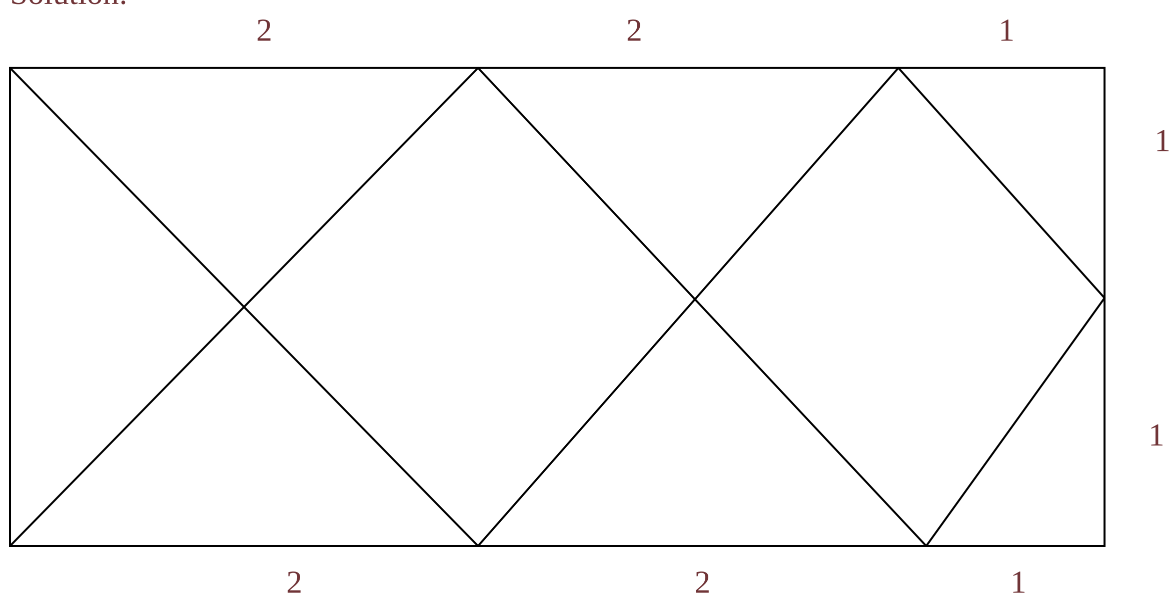
3. Stella is an excellent billiards player. If she puts the ball in one corner of a 2-foot by 5-foot billiards table and shoots it at an angle of 45° to the table, and the ball ricochets (or bounces) at a 45° angle every time it hits a side, how many times will the ball **ricochet** before it hits another corner? Count ricochets only; do not count the starting and ending points of the ball.

ANSWER: 5

Solution:



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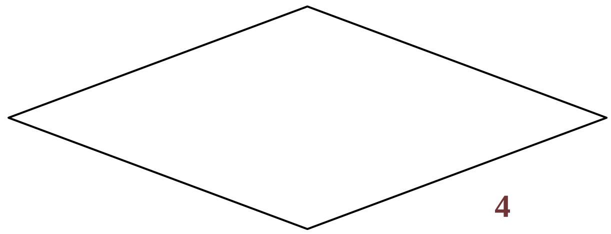
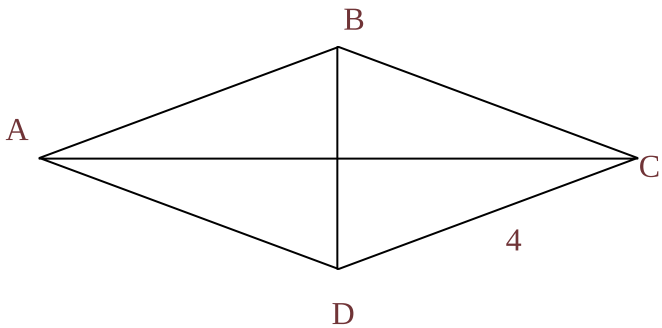
4. A rhombus has sides 4 inches long. If one pair of opposite corners are 6 inches apart, what is the **area** of the rhombus in square inches?



ANSWER: $6\sqrt{7}$

Solution: The diagonals of a rhombus divide it into 4 right triangles.



The base of one of the smaller triangles is 3, since the corners are 6 apart. The height is given by $h^2 + 3^2 = 4^2$, so $h^2 = 7$, so $h = \sqrt{7}$

The total area will be $(4)(1/2)bh$, or $2(3)\sqrt{7}$, or $6\sqrt{7}$.

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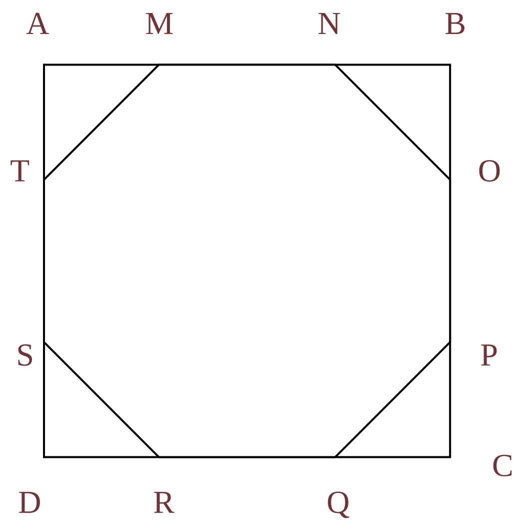
5. Find the exact value of cos 75°.

ANSWER: $\frac{\sqrt{6}-\sqrt{2}}{4}$

Solution: $\cos 75° = \cos(30° + 45°) = \cos 30° \cos 45° - \sin 30° \sin 45° = \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$.

6. In the given figure, *ABCD* is a square and *MNOPQRST* is a regular octagon with perimeter $48\sqrt{2}$ cm. Find the exact **perimeter** of square *ABCD* in cm.



ANSWER: $48 + 24\sqrt{2}$

Solution: Since the perimeter of the octagon is $48\sqrt{2}$,the length of one side must be $6\sqrt{2}$. Now, at the corners of the square are isosceles right triangles, with hypotenuse $6\sqrt{2}$. So, from the Pythagorean Theorem, $s^2 + s^2 = 72$, implying $s^2 = 36$, or $s = 6$.

So, each side of the square has length $6 + 6\sqrt{2} + 6$, or $12 + 6\sqrt{2}$. So, the total perimeter must be $48 + 24\sqrt{2}$.

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7. After the daily U.P. blizzard, Susan goes out at 4:00 P.M. to shovel the driveway. Jacqueline goes out at 4:30 P.M. to help her. If Susan can remove 20 cubic feet of snow in an hour, and Jacqueline can remove half that, at **what time** will they finish shoveling the 100 cubic feet of snow in the driveway?

ANSWER: 7:30 P.M.

Solution: Susan shovels at a rate of 20 for a time of t, so Susan shoveled a total of $20t$ feet of snow. Jacqueline shovels at a rate of 10 for a time of $t - \frac{1}{2}$, so Jacqueline shoveled a total of $10t - 5$ cubic feet of snow. Together, they shoveled 100 cubic feet, so $20t + 10t - 5 = 100$. So, $30t = 105$, so $t = \frac{105}{30} = \frac{210}{60}$ hr, or $3\ {}^{1}\!/_{2}$ hours, and, since Susan started at 4:00, they must have ended at 7:30.

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8. There are 90 students in a certain high school's graduating class.

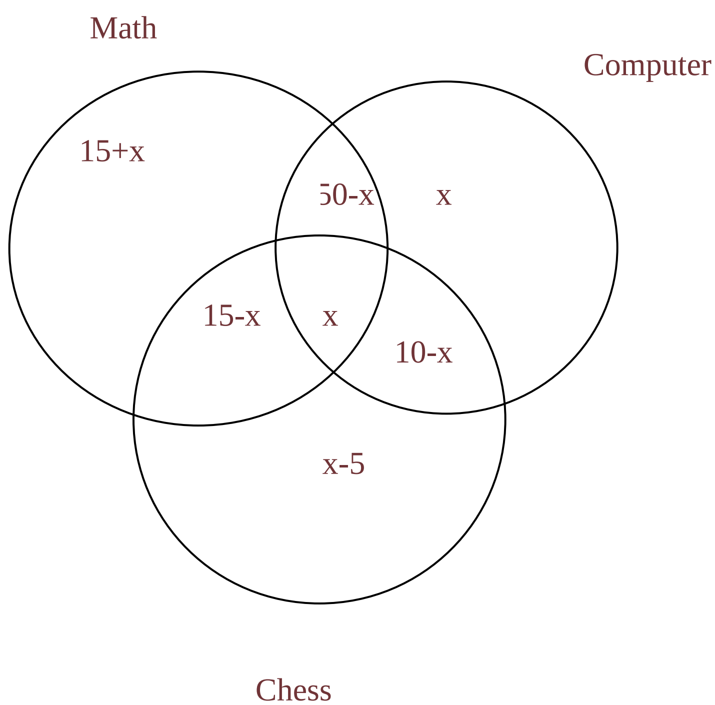
80 are in the math club.
60 are in the computer club.
20 are in the chess club.

15 are in both the math and chess clubs,
50 are in both the math and computer clubs.
10 are in both the computer and chess clubs.

If every student is in at least one club, how many students are in **all three**?

ANSWER: 5

Solution: Venn Diagrams!



Note that x is the number of students in all 3 clubs. From that, we can come up with expressions for the people in only two clubs, and from there we can get the expressions from the people in one club only.

Since there are 90 students, $15+x + 50–x + x + 15–x + x + 10–x + x–5 = 90$, so $85+x = 90$, so $x = 5$.

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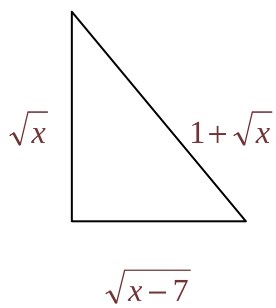
9. Given the right triangle with sides as labeled, find the length of the **hypotenuse**.



ANSWER: 5

Solution: From the Pythagorean Theorem,

$(\sqrt{x})^2 + (\sqrt{(x-7)})^2 = (1+\sqrt{x})^2$

$x + (x-7) = 1 + 2\sqrt{x} + x$

$2\sqrt{x} = (x-8)$

$4x = x^2 - 16x + 64$, squaring both sides.

$x^2 - 20x + 64=0$, so $(x-16)(x-4) = 0$, so x could be 16 or 4....except that 4 won't work since one of the sides of the triangle would be the square root of a negative number.

So, $x = 16$, and the hypotenuse has length 5.

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10. Valerie walks from her home to her school. One-fifth of the way there she finds a nickel. One-fourth of the rest of the way she finds a dime. If she is still 2 blocks from school when she finds the dime, **how many blocks** did she walk from her home to her school? (Assume that all blocks are equal in length.)

ANSWER: $3\ {}^1/_3$

Solution: If x is the total distance she walked, then$(\frac{3}{4})(\frac{4}{5})\,x = 2$. So, $(\frac{3}{5})x = 2$, so $x = \frac{10}{3}$, or $3\ {}^1/_3$.

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TEAM PROBLEMS

1. The equation of one diagonal of a rhombus is $3x + 2y = 6$. If one of the endpoints of the other diagonal has coordinates $(6, -3)$, write the equation of this other diagonal in the form $y = mx + b$.

   ANSWER: $\boldsymbol{y = \frac{2}{3}x - 7}$

   Solution: The critical thing to remember is that the diagonals of a rhombus are perpendicular to each other. And therefore their slopes are negative reciprocals of each other. $3x + 2y = 6$ is equivalent to $2y = -3x + 6$, or $y = -\frac{3}{2}x + 3$. The slope of the other diagonal must therefore be $\frac{2}{3}$.

   Our line must therefore be $y = \frac{2}{3}.x + b$. Plugging in our point, we get $-3= \frac{2}{3}.(6) + b$, making b equal to -7. The equation must be $y = \frac{2}{3}.x - 7$.

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2. What is the sum of $\frac{1}{10} + \frac{2}{100} + \frac{3}{1000} + \frac{4}{10000} + \ldots$? (An infinite series)

   ANSWER: $\boldsymbol{\frac{10}{81}}$

   Solution: Breaking it up into other series:

   0.1 + 0.01 + 0.001 + 0.0001 + 0.00001 + ... = 0.1111... = 1/9
   0.01 + 0.001 + 0.0001 + 0.00001 + ... = 0.0111 = 1/90
   0.001 + 0.0001 + 0.00001 + ... = 0.0011 = 1/900

   and so forth. This gives us another infinite geometric series on the right, with common ratio $\frac{1}{10}$, and starting term $\frac{1}{9}$. Using $S = \frac{a}{1-r}$, we have $S = \frac{1/9}{9/10} = (\frac{1}{9})(\frac{10}{9}) = \frac{10}{81}$.

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3. Factor completely: $(x^2 + 2x - 4)^2 - (x^2 - 2x - 4)^2$

   ANSWER: $\boldsymbol{8x(x-2)(x+2)}$

   Solution: From difference of two squares, we have:

$(x^2+2x-4+x^2-2x-4)(x^2+2x-4-x^2+2x+4) = (2x^2-8)(4x) = 2(4x)(x^2-4)=8x(x-2)(x+2)$

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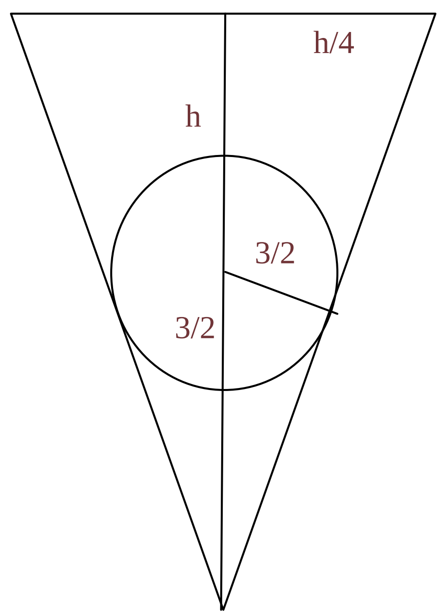
4. A giant right circular cone has its tip on the ground and its circular base in the sky. The cone is one mile high and half a mile across at its base. A ball three feet across is dropped into the cone and falls until it gets wedged. How far is the bottom of the ball from the tip of the cone at that time?

ANSWER: $\frac{3\sqrt{17}-3}{2}$

Solution:



If h is the height of the cone, and $h/4$, the radius of the circular base, the "slant-height" of the cone can be found by the Pythagorean Theorem: $h^2 + (\frac{h}{4})^2 = s^2$. So, $s = \frac{\sqrt{17}}{4}h$. So, the ratio of the slant-height to the radius of the cone is $\sqrt{17}$.

Note that the smaller right triangle in the diagram is similar to the larger one. If d is the distance from the center of the ball to the tip of the cone, $\frac{d}{3/2} = \sqrt{17}$, so $d = \frac{3}{2}\sqrt{17}$. Since the question asked for the distance between the bottom of the ball and the tip, we have to subtract the radius of the ball so the radius is $\frac{3\sqrt{17}-3}{2}$ feet.

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5. The sine of a certain angle is $\frac{3}{5}$. The cosine is $-\frac{4}{5}$. What is the tangent of twice this angle?

   ANSWER: $-\frac{24}{7}$

   Solution: $\sin 2x = 2 \sin x \cos x = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$.
   $\cos 2x = \cos^2 x - \sin^2 x = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$.

   $\tan 2x = \frac{\sin 2x}{\cos 2x} = -\frac{24}{7}$