The 58th MMPC Part II Exam and Solutions
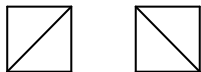
1. If P is a (convex) polygon, a triangulation of P is a set of line segments joining pairs of corners of P in such a way that P is divided into non-overlapping triangles, each of which has its corners at corners of P. For example, the following are different triangulations of a square.

    (a) Prove that if P is an n-gon with $n > 3$, then every triangulation of P produces at least two triangles T_1, T_2 such that two of the sides of T_i, $i = 1$ or 2, are also sides of P.

    (b) Find the number of different possible triangulations of a regular hexagon.

    Solution
   (a) Call the added line segments chords.
        First show by induction that each triangulation of an n-gon consists of $n - 3$ chords, which divide the n-gon into $n - 2$ triangles. This is trivially true for $n = 3$. Suppose it is true for all values of k with $3 \leq k < n$. An arbitrary chord of an n-gon divides it into a k-gon and an $n + 2 - k$-gon, since the total number of sides in the two figures is $n + 2$. By assumption the two figures have, respectively, $k - 3$ and $n - k - 1$ segments, so the total number of chords in the triangulation is $1 + (k - 3) + (n - k - 1) = n - 3$. Furthermore, the number of triangles in the two sub-figures is, respectively, $k - 2$ and $n - k$. So there are $k - 2 + n - k = n - 2$ triangles.
        The $n - 2$ triangles contain n sides of the original polygon, but none of them can contain 3 such sides. It follows that there are at least two triangles that contain 2 sides of the original polygon.

    (b) There are three types of triangulation, corresponding to the three possible patterns formed by the triangles meeting the conditions of the first part.
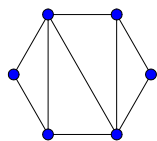   There could be two adjacent exterior triangles, two opposite exterior triangles, or three opposite triangles. The total number of different triangulations is 14.

    Case 2A The chords emanate from a single corner.
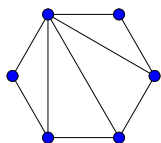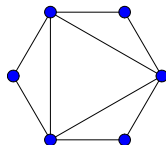
    There are six of these, one for each corner.

    Case 2O The three chords form an N-pattern (or its mirror image).

    There are 3 of these for the N and 3 for the mirror image.

    Case 3O The three chords form a triangle.

    There are two of these.

2. There are n students, $n \geq 2$, and $n + 1$ cubical cakes of volume 1. They have the use of a knife. In order to divide the cakes equitably they make cuts with the knife. Each cut divides a cake (or a piece of a cake) into two pieces.

   (a) Show that it is possible to provide each student with a volume $(n + 1)/n$ of a cake while making no more than $n - 1$ cuts.

   (b) Show that for each integer k with $2 \leq k \leq n$ it is possible to make $n - 1$ cuts in such a way that exactly k of the n students receive an entire (uncut) cake in their portion.

   Solution
   (a) Give one complete cake to each student and divide the last one in n equal pieces with $n - 1$ cuts.

   (b) Clearly $k = n$ can be achieved by giving one complete cake to each student and by dividing the $(n + 1)$th one in n equal pieces with $n - 1$ cuts. Let $k \in \{2, 3, \ldots, n - 1\}$. Give one entire cake to Student 1, Student 2, up to Student $(k - 1)$. Each of them needs an additional $1/n$ of a cake. We have $n - k + 2$ cakes left to distribute. Line up the cake from left to right and pretend that it is one long piece of rectangular solid. Make $n - k + 1$ "cuts" to produce $n - k + 1$ shares of $(n + 1)/n$ of a cake to give to Student k, Student $k + 1$, up to Student n, each having his/her proper share. The leftover piece is $(k - 1)/n$ of a cake. With $k - 2$ cuts, it can be divided into $k - 1$ pieces of $1/n$ of a cake. We distribute them to Student 1, Student 2, up to Student $k - 1$. The total number of "cuts" is $(n - k + 1) + (k - 2) = n - 1$. Note that by construction, Student k also receive an entire cake in his/her share. We need to show that none of Student $k + 1$, Student $k + 2$, up to Student n receives an entire cake. Note also that cuts are in "quotes" in the construction step as we have not eliminated the possibility that a "cut" is at the joint of two cakes, making such a cut unnecessary. We will now show that this is impossible, which also implies that none of Student $k + 1$, Student $k + 2$, up to Student n receives a complete cake. If the ith "cut" is at a joint, then $i(1 + 1/n)$ is an integer, which is impossible as $i < n$.

3. The vertical lines at $x = 0$, $x = \frac{1}{2}$, $x = 1$, $x = \frac{3}{2}$, ... and the horizontal lines at $y = 0$, $y = \frac{1}{2}$, $y = 1$, $y = \frac{3}{2}$, ... subdivide the first quadrant of the plane into $\frac{1}{2} \times \frac{1}{2}$ square regions. Color these regions in a checkerboard fashion starting with a black region near the origin and alternating black and white both horizontally and vertically.

(a) Let T be a rectangle in the first quadrant with sides parallel to the axes. If the width of T is an integer, prove that T has equal areas of black and white. Note that a similar argument works to show that if the height of T is an integer, then T has equal areas of black and white.

(b) Let R be a rectangle with vertices at $(0,0)$, $(a,0)$, (a,b), and $(0,b)$ with a and b positive. If R has equal areas of black and white, prove that either a is an integer or that b is an integer.

(c) Suppose a rectangle R is tiled by a finite number of rectangular tiles. That is, the rectangular tiles completely cover R but intersect only along their edges. If each of the tiles has at least one integer side, prove that R has at least one integer side.

Solution

(a) The grid subdivides the rectangle T into black rectangles and white rectangles. Consider a horizontal row of these smaller rectangles across T. Let B be the line segment along the bases of this row of rectangles. Since the length of B is an integer, B can be broken up into segments of unit length.

Each of these unit-length segments will be a portion of the base of some length x (with $0 < x \le \frac{1}{2}$) of a rectangle of one color, the base of length $\frac{1}{2}$ of a rectangle of the other color, and a portion of the base of length $\frac{1}{2} - x$ of a rectangle of the first color.

Thus the length of the portions of B that are the bases of black rectangles is equal to the length of the portions that are the bases of white rectangles.

Since the heights of these rectangles are all the same, the areas of the black rectangles above B is equal to the areas of the white rectangles above B. This is true for any horizontal row of rectangles across T. It follows that T has equal areas of black and white.

(b) Suppose that neither a nor b is an integer. Let $[a]$ denote the greatest integer that is less than or equal to a. Likewise, let $[b]$ denote the greatest integer that is less than or equal to b. The rectangle with vertices $(0,0)$, $([a],0)$, $([a],b)$, and $(0,b)$ has integer width. By part 3a, it has equal areas of black and white. The rectangle with vertices $([a],0)$, $(a,0)$, and $(a,[b])$ and $([a],[b])$ has integer height. By part 3a, it also has equal areas of black and white.

Consider the remaining portion of T in the upper right corner. Let $x = a - [a] - \frac{1}{2}$ and let $y = b - [b] - \frac{1}{2}$. Notice that $-\frac{1}{2} < x < \frac{1}{2}$ and $-\frac{1}{2} < y < \frac{1}{2}$.

If $x < 0$ and $y < 0$, this will be a black rectangle of positive area $(a - [a])(b - [b])$. If $x > 0$ and $y < 0$, this will be a black rectangle of area $\frac{1}{2}(b - [b])$ and a white rectangle of smaller area $x(b - [b])$. Likewise, if $x < 0$ and $y > 0$, this will be a black rectangle of area $\frac{1}{2}(a - [a])$ and a white rectangle of smaller area $y(a - [a])$.

Finally, if $x > 0$ and $y > 0$, this will be two black rectangles of areas $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and xy and two white rectangles of areas $\frac{1}{2}x$ and $\frac{1}{2}y$. Since $0 < (x - \frac{1}{2})(y - \frac{1}{2}) = xy - \frac{1}{2}x - \frac{1}{2}y + \frac{1}{4}$, it follows that the black area of $xy + \frac{1}{4}$ will exceed the white area of $\frac{1}{2}x + \frac{1}{2}y$.

Thus, only when a is an integer (and $a = [a]$) or b is an integer (and $b = [b]$) will T have equal areas of black and white.

(c) Position R in the first quadrant with vertices at $(0,0)$, $(a,0)$, (a,b), and $(0,b)$. By part a, each of the tiles has equal areas of black and white. Thus R has equal areas of black and white. By part b, the length of one of the sides of R must be an integer.

4. Call a number *simple* if it can be expressed as a product of single-digit numbers (in base ten).

   (a) Find two simple numbers whose sum is 2014 or prove that no such numbers exist.

   (b) Find a simple number whose last two digits are 37 or prove that no such number exists.

   Solution
   (a) There are many answers, for example, $2000 + 14$.

   (b) There is no such number. The only possible prime factors a simple number can have are 2, 3, 5, and 7. If a number is divisible by 2 or 5, then its last digit will not be 7. So the only numbers we need to consider have the form $N = 3^a 7^b$. Now consider $N \bmod 20$. Modulo 20, the powers of 3 are 3, 9, 7, 1. Likewise the powers of 7 are 7, 9, 3, 1. The product of any numbers in the set $\{1, 3, 7, 9\}$ is, modulo 20, again a number in this set. However, any number whose last digits are 37 will be 17 modulo 20.

5. Consider triangles for which the angles α, β, and γ form an arithmetic progression. Let a, b, c denote the lengths of the sides opposite α, β, γ, respectively. Show that for all such triangles,
$$\frac{a}{c} \sin 2\gamma + \frac{c}{a} \sin 2\alpha$$
has the same value, and determine an algebraic expression for this value.

   Solution
   Let d denote the common difference between terms in the arithmetic sequence α, β, and γ. Then $180° = \alpha + \beta + \gamma = (\beta - d) + \beta + (\beta + d) = 3\beta$. So $\beta = 60°$.
   Now
$$
\begin{aligned}
\frac{a}{c} \sin 2\gamma + \frac{c}{a} \sin 2\alpha &= \frac{2a \sin \gamma \cos \gamma}{c} + \frac{2c \sin \alpha \cos \alpha}{a} && \text{(double angle formula)} \\
&= \frac{2a \sin \alpha \cos \gamma}{a} + \frac{2c \sin \gamma \cos \alpha}{c} && \text{(law of sines)} \\
&= 2(\sin \alpha \cos \gamma + \sin \gamma \cos \alpha) \\
&= 2 \sin(\alpha + \gamma) && \text{(angle addition formula)} \\
&= 2 \sin 2\beta \\
&= 2 \sin 120° \\
&= \sqrt{3}.
\end{aligned}
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